DIFFERENCE AND DIFFERENTIAL EQUATIONS

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Reference material

For difference equations:

J. D. Hamilton (1994). *Time Series Analysis*. Princeton University Press.

For differential equations:

S. G. Krantz (2014). *Differential Equations: Theory, Technique and Practice*. Chapman and Hall/CRC.

Motivating examples

Consider a loan taken out at constant interest rate r.

Let $D(t)$ be debt outstanding at beginning of period t and let $Z(t)$ be the repayment at the end of the period.

Initial debt level is $D(0)$.

Debt evolves as

$$D(t+1) = D(t) + rD(t) - Z(t) = (1+r)D(t) - Z(t).$$

This is a difference equation.

Time here is discrete.

Similar example is interest accumulation:

$$D(t+1) = D(t) + rD(t) = (1+r)\,D(t).$$

By backward substitution,

$$D(t+h) = (1+r)^h D(t) = \big(1 + \log(1+r)\,h + O(h^2)\big)\ D(t),$$

where $O(h^2)$ collects terms that shrink to zero at least as fast as h^2 when $h \to 0$. Use a Taylor series of $a^h = e^{\log(a^h)} = e^{h\ \log(a)}$ around $h = 0$ to get this:

$$e^{h\log(a)} = e^{0\log(a)} + e^{0\log(a)}\log(a)h + \frac{1}{2}e^{0\log(a)}\log(a)^2h^2 + \cdots = 1 + \log(a)h + \cdots$$

Now re-arrange to get

$$\frac{D(t+h) - D(t)}{h} = (\log(1+r) + O(h))\ D(t).$$

Taking limits as $h \to 0$ gives

$$\frac{dD(t)}{dt} = \log(1+r)\,D(t).$$

This is a differential equation. Time is continuous.

Population size at time t is $P(t)$ and evolves like

$$\frac{dP(t)}{dt} = a\,P(t)\,\left(1 - \frac{1}{c}P(t)\right),$$

where $c > 0$.

Suppose that $c = \infty$, so that

$$\frac{dP(t)}{dt} = a\,P(t).$$

This is called the Malthusian model.

This model implies that in the long run population grows exponentially.

The introduction of finite c imposes a capacity constraint. This is called the Verhulst model.

When population is small the correction factor is small. When population grows large, the correction term (which is quadratic) starts to dominate the Malthusian term (which is linear).

Outline: Differential equations

- First-order differential equations
  - Existence of a solution
  - Solutions methods and examples
  - Direction fields and phase diagrams
  - Application: Solow's growth model
- Numerical approximation
- Second-order equations
  - homogeneous case
  - Non-homogeneous case
- Systems of (first-order) equations
  - homogeneous case
  - Non-homogeneous case
  - Phase planes

Differential equations

Let $y = y(x)$ be a differentiable function.

In a typical application x will be time or distance.

We write derivatives as $y' = dy/dx$, $y'' = d^2y/dx^2$, and so on.

A differential equation relates y to one or more of its derivatives.

An nth-order equation involves up to the nth-derivative. For a function F,

$$y' = F(x, y)$$

is a first-order equation.

A first-order linear equation is

$$y' + a(x)y = b(x)$$

for functions a and b.

The goal is to recover the function(s) y.

An example

A simple first-order linear equation is

$$y' + ay = 0$$

for a constant a.

This is a homogeneous first-order equation with constant coefficient.

An educated guess shows that

$$y = e^{-ax}$$

is a solution to this differential equation.

Indeed, $y' = -ae^{-ax}$ and so

$$y' + ay = -ae^{-ax} + ae^{-ax} = 0.$$

The family of curves $y = ce^{-ax}$ for constants c collects all solutions.

We are generally not interested in the trivial solution $y = 0$.

If we complement the equation with an initial condition a single member of the family can be selected as the solution.

Suppose that $y(x_0) = y_0$. Then

$$y_0 = ce^{-ax_0}$$

follows from the general solution.

Consequently, we have that $c = y_0 e^{ax_0}$ and

$$y = ce^{-ax} = \left(y_0 e^{ax_0}\right)\left(e^{-ax}\right) = y_0 e^{-a(x-x_0)}$$

is the solution to this initial-condition problem.

Picard theorem

Consider

$$y' = F(x, y(x)).$$

Suppose that F is

a) continuous in x,

b) Lipschitz in y, i.e.,

$$|F(x, s) - F(x, q)| \leq c\,|s - q|$$

for finite constant c (independent of x).

Then, for a neighborhood around x_0, there exists a unique function y for which

$$y' = F(x, y(x)), \qquad y(x_0) = y_0,$$

for all x in that neighborhood.

Solutions, approximations, and direction fields

Picard's result guarantees that a well-defined solution exists.

This does not imply that an explicit solution is always available.

For several classes of differential equations we do have explicit solutions but for many others we do not.

When such expressions are lacking we can resort to numerical approximation methods.

Direction fields and phase diagrams are also useful to understand the behavior of solutions.

First-order linear equations

The general solution to

$$y' + a(x)y = b(x)$$

can be found in several ways.

One way that always works is to multiply through by the integrating factor

$$A(x) = e^{\int a(x)\,dx}.$$

Note that $A' = a\,A$.

Multiplying through the equation yields $A(x)y' + A(x)a(x)y = A(x)b(x)$.

By the chain rule, this is

$$(A(x)y)' = A(x)b(x),$$

and integration yields the solution.

An example

In

$$y' + 2xy = x$$

the integrating factor is e^{x^2}.

Multiplication yields

$$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}x.$$

Thus, $(e^{x^2}y)' = e^{x^2}x$ and

$$e^{x^2}y = \frac{1}{2}e^{x^2} + c$$

follows by integration.

Re-arranging yields the explicit solution family

$$y = \frac{1}{2} + ce^{-x^2}$$

An alternative derivation

For the equation

$$y' + a(x)y = b(x)$$

we call the homogeneous version

$$y' + a(x)y = 0$$

the complementary equation.

The complementary equation, we know, can be solved by integrating the re-arranged equation

$$\frac{y'}{y} = -a(x),$$

which yields $\log|y| = -\int a(x)dx$ or $y = ce^{-\int a(x)dx}$. Let y_1 denote one of these solutions.

Now we look for solutions of the form $y = uy_1$ for the non-homogeneous equation.

By the chain rule

$$(uy_1)' = u'y_1 + uy_1',$$

and so by substitution the original equation at $y = uy_1$ can be written as

$$(u'y_1 + uy_1') + a(x)uy_1 = b(x).$$

Now re-arrange to write

$$u'y_1 + u(y_1' + a(x)y_1) = b(x)$$

and exploit the fact that $y_1' + a(x)y_1 = 0$, by construction, to get $u'y_1 = b(x)$, and thus

$$u' = \frac{b(x)}{y_1(x)}.$$

We find u and thus $y = uy_1$ by integration.

An example

For the example

$$y' + 2xy = x$$

the complementary equation is $y_1' + 2xy_1 = 0$.

We know that

$$y_1 = e^{-x^2}$$

is a particular solution.

Thus,

$$u = \int \frac{x}{e^{-x^2}} dx = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + c.$$

Therefore,

$$y = uy_1 = \left(\frac{1}{2}e^{x^2} + c\right) e^{-x^2} = \frac{1}{2} + ce^{-x^2},$$

which, of course, agrees with our result from before.

More examples

$$xy' + \log(x)y = 0$$

$$y' + 3y = x$$

Separable equations

A (nonlinear) separable equation is of the form

$$h(y)y' = g(x)$$

for (integrable) functions h and g.

Let $H(y) = \int h(y)dy$ and $G(x) = \int g(x)dx$.

By the chain rule

$$\frac{dH(y(x))}{dx} = \frac{dH}{dy}\frac{dy}{dx} = h(y)y'.$$

Our original equation can thus be written as

$$\frac{dH(y(x))}{dx} = \frac{dG(x)}{dx}$$

and integration gives

$$H(y) = G(x) + c$$

which yields an implicit solution to the differential equation.

An example

A nonlinear equation is

$$y' = -\frac{x}{y}.$$

Re-arrange to get

$$yy' = -x,$$

verifying that this is separable with $h(y) = y$ and $g(x) = -x$.

The implicit solution is

$$\frac{y^2 + x^2}{2} = c.$$

(note that $c < 0$ make no sense here)

We can re-define the integration constant and write $y^2 + x^2 = c^2$, which can be solved for y to get

$$y = \begin{cases} \sqrt{c^2 - x^2} & \text{for } -c < x < c \\ -\sqrt{c^2 - x^2} & \text{for } -c < x < c \end{cases}$$

These are semi-circles (with radius c) above and below the horizontal axis.

Another example

Standard logistic equation is (with $y \in (0,1)$)

$$y' = y(1-y)$$

and is separable as

$$\frac{1}{y(1-y)} y' = 1.$$

Note that

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

and so

$$\int \frac{1}{y(1-y)}\, dy = \log y - \log(1-y).$$

Hence,

$$y = \frac{ce^x}{1+ce^x}.$$

With an initial condition we have

$$y_0 = \frac{ce^{x_0}}{1 + ce^{x_0}}$$

which gives

$$c = \frac{y_0}{(1 - y_0)} e^{-x_0}$$

and so

$$y = \frac{\frac{y_0}{(1-y_0)} e^{-x_0} e^{x}}{1 + \frac{y_0}{(1-y_0)} e^{-x_0} e^{x}} = \frac{e^{x-x_0}}{\frac{1-y_0}{y_0} + e^{x-x_0}} = \frac{y_0}{y_0 + (1 - y_0)e^{-(x-x_0)}}$$

is the specific solution.

More examples

$x^5 y' + y^5 = 0$

$$yy' = x + 1$$

$y' = x^2 y^2$

Direction fields

For the equation

$$y' = F(x, y)$$

an explicit solution is typically not available.

Nonetheless, for any given point (x, y) in the Cartesian plane we can calculate the slope of the solution, $F(x, y)$.

A direction field plots the slope on a grid of such points.

Consider

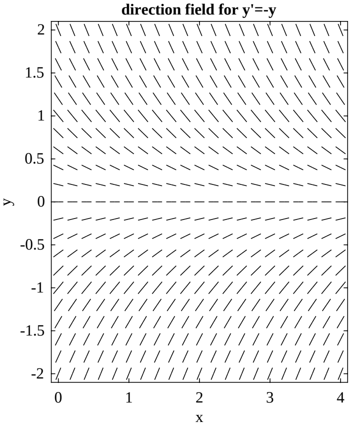
$$y' = -y,$$

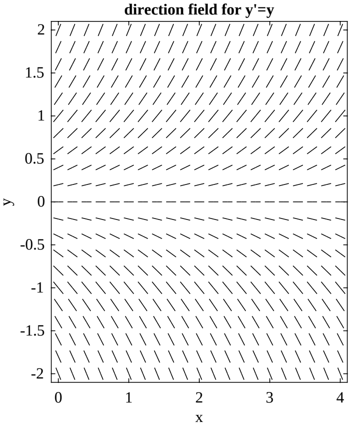
with general solution (from above) given by $y = ce^{-x}$.

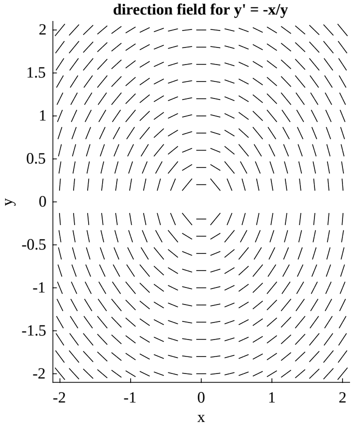
We have

$$\begin{array}{ll} y' < 0 & \text{if } y > 0 \\ y' = 0 & \text{if } y = 0 \\ y' > 0 & \text{if } y < 0 \end{array},$$

independent of x. Further, $|y'|$ decreases as $|y|$ increases, converging to zero.


direction field for y'=-y
y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4


direction field for y'=y
y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4


direction field for y' = -x/y
direction field for y' = x/y
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1
0
1
2
x

Phase diagram

Equations of the form

$$y' = F(y)$$

are called autonomous.

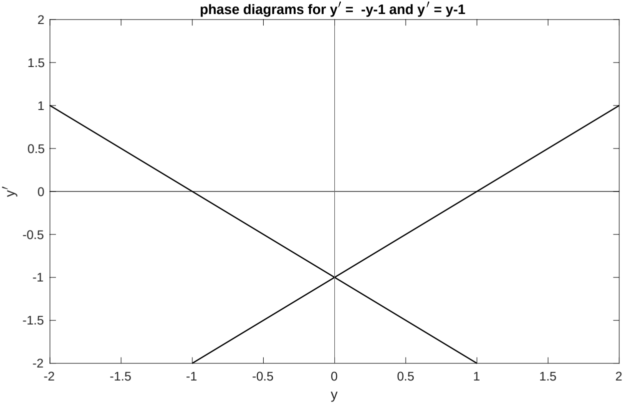
We can plot y' against y to assess the behavior of the solution.

An equilibrium y^* is any point such that

$$F(y^*) = 0$$

(the slope is zero at such a point).

An equilibrium y^* is (locally) stable if $F'(y^*) < 0$ and unstable if $F'(y^*) > 0$. Otherwise it is semi-stable.



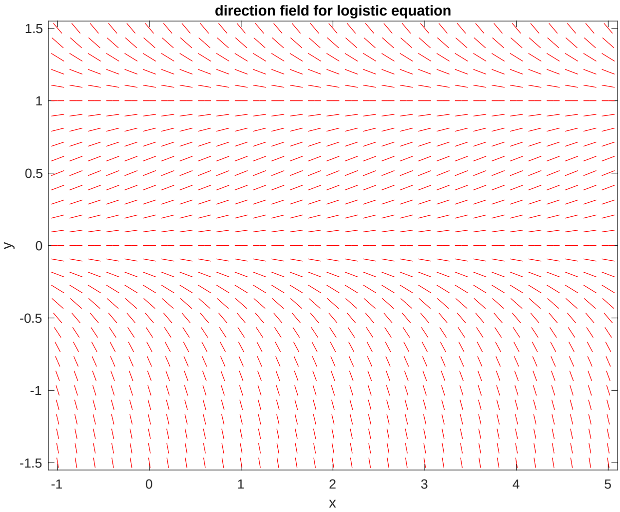
phase diagrams for y′ = -y-1 and y′ = y-1
y′
y
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

An example

Draw the phase diagram for the logistic equation

$$y' = y(1 - y).$$

Give the equilibrium point(s) and classify them as stable/unstable.


direction field for logistic equation
1.5
1
0.5
0
-0.5
-1
-1.5
y
-1
0
1
2
3
4
5
x

An application

Output is produced using capital K and labor L according to

$$AK^{1-a}L^a, \qquad A > 0, \qquad a \in (0, 1).$$

A (constant) fraction s of output is saved an re-invested in capital while labor grows at constant rate q. We begin with initial stocks K_0, L_0 at time zero. This is Solow's growth model.

So,

$$K' = sAK^{1-a}L^a, \qquad L' = qL.$$

Clearly, the general solution for labor is $L = ce^{qx}$ and so

$$L = L_0 e^{qx}.$$

Substitution gives $K' = sAK^{1-a}L_0^a e^{aqx}$ and yields the separable equation

$$K^{a-1}K' = sAL_0^a e^{aqx}.$$

Hence,

$$K = \sqrt[a]{\frac{sAL_0^a}{q} e^{aqx} + c}.$$

The initial condition gives $c = K_0^a - sAL_0^a/q$ and so

$$K = \sqrt[a]{sAL_0^a \frac{e^{aqx} - 1}{q} + K_0^a}.$$

The capital to labor ratio is

$$k = \frac{K}{L} = \sqrt[a]{\frac{sAL_0^a \frac{e^{aqx}-1}{q} + K_0^a}{L_0^a e^{aqx}}}$$

$$= \sqrt[a]{\frac{s}{q}\left(A - e^{-aqx}\right) + \left(\frac{K_0}{L_0}\right)^a e^{-aqx}} \longrightarrow \sqrt[a]{\frac{s}{q} A}$$

as $x \uparrow \infty$.

Use the shorthand $f(k) = Ak^{1-a}$.

We have

$$k' = \left(\frac{K}{L}\right)' = \frac{K'L - KL'}{L^2} = \frac{K' - kL'}{L}$$

and so

$$k' = sf(k) - qk,$$

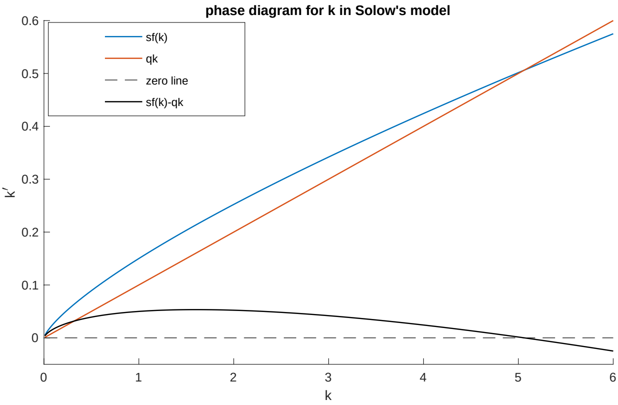
which is nonlinear but autonomous.

The steady state of k is found by solving

$$sf(k) - qk = 0$$

for k and yields

$$k^* = \sqrt[a]{\frac{s}{q}A}.$$


phase diagram for k in Solow's model
sf(k)
qk
zero line
sf(k)-qk
0.6
0.5
0.4
0.3
0.2
0.1
0
k′
0
1
2
3
4
5
6
k

Numerical approximation

When a closed-form solution is not available we may compute a numerical approximation to it.

A simple such approximating function can be obtained using Euler's method.

Say that

$$y'(x) = F(x, y(x)), \qquad y(x_0) = y_0.$$

Recall that

$$y(x) = \int_{-\infty}^{x} y'(t)\,dt = \int_{-\infty}^{x} F(t, y(t))\,dt$$

by the fundamental theorem of calculus.

Hence, for two points $x_0 < x_1$,

$$y(x_1) - y(x_0) = \int_{-\infty}^{x_1} F(t, y(t))\,dt - \int_{-\infty}^{x_0} F(t, y(t))\,dt = \int_{x_0}^{x_1} F(t, y(t))\,dt,$$

or,

$$y(x_1) = y_0 + \int_{x_0}^{x_1} F(t, y(t))\,dt.$$

In

$$y(x_1) = y_0 + \int_{x_0}^{x_1} F(t, y(t))\, dt$$

we do not know $y(t)$ for $t \in (x_0, x_1)$.

Now let $x_1 = x_0 + h$ for a small $h > 0$.

Assuming that $y(x_0 + h) \approx y(x_0)$ for small h an approximation to $y(x_1)$ is

$$y_1 = y_0 + \int_{x_0}^{x_1} F(x_0, y_0)\, dt = y_0 + h\, F(x_0, y_0),$$

which can be computed.

The iteration

$$y_{k+1} = y_k + h\, F(x_k, y_k)$$

then suggests itself.

Clearly, the error gets worse for larger k and, all else equal, for larger h.

Order reduction

First-order equations are central.

Higher-order differential equations can be reduced to (a system of) first-order equations.

Consider the second-order linear equation

$$y'' + a(x)y' = b(x).$$

Note that the level y does not feature here.

We can perform the substitution $y' = p$. Then $y'' = p'$ and so

$$p' + a(x)p = b(x)$$

is a first-order equation.

We can first solve for p and then for y in a second step.

We should expect two free parameters here, and so two initial conditions will be needed to pin both of them down.

An example

Say that

$$xy'' - y' = 3x^2.$$

Substitution and re-arrangement gives

$$p' - \frac{1}{x}p = 3x.$$

Using integrating factor $e^{\int -x^{-1}dx} = e^{-\log(x)} = e^{\log(x^{-1})} = x^{-1}$ its solution is

$$p = 3x^2 + cx.$$

Recalling that $p = y'$ we have

$$y' = 3x^2 + cx,$$

which is a polynomial; so

$$y = x^3 + \frac{c}{2}x^2 + d$$

is the solution to the original second-order equation.

A second-order equation

In the homogeneous constant-coefficient problem

$$ay'' + by' + cy = 0$$

a natural candidate solution is of the form $y = e^{rx}$. Indeed, for this solution,

$$a\left(e^{rx}\right)'' + b\left(e^{rx}\right)' + c\left(e^{rx}\right) = 0$$

yields $ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$ but for this to be true at all x we will need that

$$ar^2 + br + c = 0$$

for which the solutions are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This means we have, in general, two solutions of the form e^{r_1x} and e^{r_2x}, and so

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

as general solution.

An example

Consider the problem

$$y'' + 6y' + 5y = 0$$

subject to $y(0) = 3$ and $y'(0) = -1$.

It has the polynomial equation

$$1r^2 + 6r + 5 = (r+1)(r+5) = 0$$

and so the roots are $r_1 = -1$ and $r_2 = -5$. The general solution follows as

$$y = c_1 e^{-x} + c_2 e^{-5x}.$$

Now

$$\begin{aligned} y(0) &= c_1 + \ \ c_2 = 3 \\ y'(0) &= c_1 + 5c_2 = 1' \end{aligned}$$

so that $c_1 = \frac{7}{2}$ and $c_2 = -\frac{1}{2}$ and

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Fundamental solution set

A fundamental solution set (y_1, y_2) to the second-order equation is one so that any solution y can be written as

$$y = c_1 y_1 + c_2 y_2,$$

which is the general solution.

y_1, y_2 are fundamental if they are linearly independent functions, i.e., they are not constant multiples of one another.

Can be re-stated as the requirement that the Wronskian (function)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

has no zeros.

In general, with two solutions, $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$

$$\frac{y_2}{y_1} = e^{(r_2 - r_1)x}$$

is non-constant (in x) if the roots are distinct.

Another example

Now take

$$y'' + 6y' + 9y = 0.$$

Then the polynomial $r^2 + 6r + 9 = (r + 3)^2$ has only one distinct root, with associated solution

$$y_1 = e^{-3x}.$$

We need a second (linearly independent) solution. We try $y = uy_1 = ue^{-3x}$. Then

$$y' = u'y_1 + uy_1' = (u' - 3u)e^{-3x}, \qquad y'' = (u'' - 6u' + 9u)e^{-3x},$$

so that

$$y'' + 6y' + 9y = u''e^{-3x} = 0$$

requires that $u'' = 0$; thus, u should be a linear function $u = c_1 + c_2x$.

The solution is $y = (c_1 + c_2x)e^{-3x}$. This is the general solution as it is a linear combination of $y_1 = e^{-3x}$ and xe^{-3x}, which are linearly independent.

Now suppose that

$$y'' + ay' + by = c.$$

We know how to obtain fundamental solutions, y_1 and y_2 to the associated homogeneous system.

We look for a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2$$

for functions u_1, u_2.

First we have

$$y_p' = (u_1' y_1 + u_1 y_1') + (u_2' y_2 + u_2 y_2') = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

and to simplify matters we enforce that $u_1' y_1 + u_2' y_2 = 0$; then

$$\begin{aligned} y_p' &= (u_1 y_1' + u_2 y_2') \\ y_p'' &= (u_1 y_1'' + u_2 y_2'') + (u_1' y_1' + u_2' y_2'). \end{aligned}$$

Now we plug these two expressions back into our differential equation to get

$$u_1\left(y_1'' + ay_1' + by_1\right) + u_2\left(y_2'' + ay_2' + by_2\right) + \left(u_1'y_1' + u_2'y_2'\right) = c(x).$$

But,

$$\begin{aligned} y_1'' + ay_1' + by_1 &= 0 \\ y_2'' + ay_2' + by_2 &= 0 \end{aligned}$$

by construction the differential equation simplifies to just

$$u_1'y_1' + u_2'y_2' = c(x).$$

Together with the condition enforced on the functions u_1, u_2 we now have a system of two equations in two unknowns:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= c(x). \end{aligned}$$

These can be solved for u_1', u_2' (which may then be integrated to yield y_p).

The general solution then becomes

$$c_1y_1 + c_2y_2 + y_p.$$

An example

Solve

$$y'' - y' - 2y = 4x^2.$$

The complementary equation is

$$y'' - y' - 2y = 0.$$

We solve

$$r^2 - r - 2 = 0$$

for r to find $r_1 = 2$ and $r_2 = -1$. This yields the general solution

$$c_1 e^{2x} + c_2 e^{-x}.$$

Now we look for a particular solution for the non-homogeneous system of the form

$$y_p = u_1 y_1 + u_2 y_2 = u_1 e^{2x} + u_2 e^{-x}.$$

For this we solve the system

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= c \end{aligned}, \qquad \text{i.e.,} \qquad \begin{aligned} u_1' e^{2x} + u_2' e^{-x} &= 0 \\ 2u_1' e^{2x} - u_2' e^{-x} &= 4x^2 \end{aligned}.$$

In matrix notation:

$$\begin{pmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 4x^2 \end{pmatrix}$$

with solution

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = -\frac{1}{3e^x} \begin{pmatrix} -e^{-x} & -e^{-x} \\ -2e^{2x} & e^{2x} \end{pmatrix} \begin{pmatrix} 0 \\ 4x^2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} x^2 e^{-2x} \\ -\frac{4}{3} x^2 e^x \end{pmatrix}.$$

Then (integrate by parts repeatedly)

$$\begin{aligned} u_1 &= -\frac{2}{3} x^2 e^{-2x} - \frac{2}{3} x e^{-2x} - \frac{1}{3} e^{-2x} \\ u_2 &= -\frac{4}{3} x^2 e^x + \frac{8}{3} x e^x - \frac{8}{3} e^x \end{aligned}.$$

The particular solution is then

$$y_p = -\frac{6}{3} x^2 + \frac{6}{3} x - \frac{9}{3} = -2x^2 + 2x - 3.$$

For an n-vector $\boldsymbol{y} = (y_1, \dots, y_n)^\top$ a first-order linear system of differential equations (with constant coefficients) is

$$\boldsymbol{y}' - \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b},$$

where $\boldsymbol{y}' = (y_1', \dots, y_n')^\top$.

Spelled-out this is

$$\begin{aligned} y_1' &= a_{11}y_1 + \cdots + a_{1n}y_n + b_1 \\ y_2' &= a_{21}y_1 + \cdots + a_{2n}y_n + b_2 \\ &\vdots \\ y_n' &= a_{n1}y_1 + \cdots + a_{nn}y_n + b_n \end{aligned}$$

We will mostly set $n = 2$, which is without loss of generality.

Fundamental solution set

The general solution will be of the form

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + \cdots c_n\boldsymbol{y}_n,$$

where $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ are particular, linearly-independent solutions.

Linear independence is again checked by calculating the Wronskian. Here, it equals

$$W = |\boldsymbol{Y}| = |\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n| = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix},$$

which should have no zeros.

A special case

To motivate a future development suppose that

$$\begin{aligned} y_1' &= a_{11} y_1 \\ y_2' &= a_{22} y_2 \end{aligned},$$

that is the 2-by-2 matrix $\boldsymbol{A}$ is diagonal. These equations are effectively stand-alone first-order linear equations.

The general solutions are

$$y_1 = c_1 e^{a_{11} x}, \qquad y_2 = c_2 e^{a_{22} x},$$

respectively.

The general solution to the system is

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{a_{11} x} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{a_{22} x}.$$

Now consider

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$$

for non-diagonal matrix $\boldsymbol{A}$.

Suppose that n-by-n matrix $\boldsymbol{A}$ has n linearly-independent eigenvectors.

We can factorize

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1},$$

where $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n)$ collects the eigenvectors and $\boldsymbol{D} = \text{diag}(d_1, \ldots, d_n)$ are the associated eigenvalues.

Note that $\boldsymbol{P}^{-1}\boldsymbol{A} = \boldsymbol{D}\boldsymbol{P}^{-1}$. Defining $\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{y}$ we have that

$$\boldsymbol{z}' = \boldsymbol{D}\boldsymbol{z}$$

is a system of stand-alone equations (in the variable $\boldsymbol{z}$, and not in the original $\boldsymbol{y}$, of course!).

Clearly, the solution here is

$$\boldsymbol{z} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{d_1 x} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} e^{d_2 x} + \cdots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} e^{d_n x} = \sum_{i=1}^{n} c_i \boldsymbol{e}_i e^{d_i x}$$

where we write $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ for the standard basis vectors for $\mathbb{R}^n$.

Recalling that $\boldsymbol{P}\boldsymbol{z} = \boldsymbol{y}$ we premultiply by the eigenvectors to obtain the general solution

$$\boldsymbol{y} = c_1 \boldsymbol{p}_1 e^{d_1 x} + c_2 \boldsymbol{p}_2 e^{d_2 x} + \cdots + c_n \boldsymbol{p}_n e^{d_n x} = \sum_{i=1}^{n} c_i \boldsymbol{p}_i e^{d_i x}$$

because $\boldsymbol{P}\boldsymbol{e}_i = \boldsymbol{p}_i$.

An example

In

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= \quad 6y_1 + 5y_2 \end{aligned},$$

the coefficient matrix is

$$\boldsymbol{A} = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}.$$

The eigenvalues are the solutions to

$$\begin{vmatrix} -(4+d) & -3 \\ 6 & (5-d) \end{vmatrix} = 18 - (4+d)(5-d) = (d-2)(d+1) = 0,$$

and equal $d_1 = 2$ and $d_2 = -1$.

The associated eigenvectors are, for $d_1 = 2$,

$$\begin{pmatrix} -6 & -3 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \boldsymbol{p}_1$$

and, for $d_2 = -1$,

$$\begin{pmatrix} -3 & -3 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boldsymbol{p}_2$$

(Recall that eigenvectors are up to sign and magnitude.)

This gives

$$\boldsymbol{y} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2x} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x}$$

as the general solution.

To verify, calculate the Wronskian:

$$W = \begin{vmatrix} 1e^{2x} & 1e^{-x} \\ -2e^{2x} & -1e^{-x} \end{vmatrix} = -e^{2x}e^{-x} + 2e^{2x}e^{-x} = e^{x}.$$

This is indeed never zero.

An example

Suppose that

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 11 & -25 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Here,

$$\begin{vmatrix} 11-d & -25 \\ 4 & -9-d \end{vmatrix} = (d-1)^2$$

has only one solution, $d = 1$. $\boldsymbol{A}$ cannot be diagonalized; the associated eigenvector is $\boldsymbol{p} = (5,2)^\top$.

We cannot achieve exact uncoupling of the two equations. We can, however, search for a generalized eigenvector $\boldsymbol{p}^*$ so that

$$\begin{pmatrix} 11 & -25 \\ 4 & -9 \end{pmatrix} (\boldsymbol{p}, \boldsymbol{p}^*) = (\boldsymbol{p}, \boldsymbol{p}^*) \begin{pmatrix} d & 1 \\ 0 & d \end{pmatrix};$$

in this case the reparametrized equations are triangular so we can solve them sequentially.

Note that $\boldsymbol{A}\boldsymbol{p} = d\boldsymbol{p}$ and $\boldsymbol{A}\boldsymbol{p}^* = \boldsymbol{p} + d\boldsymbol{p}^*$ so we solve $(\boldsymbol{A} - d\boldsymbol{I}_2)\boldsymbol{p}^* = \boldsymbol{p}$. to find $\boldsymbol{p}^*$.

Solving

$$\begin{pmatrix} 10 & -25 \\ 4 & -10 \end{pmatrix} \boldsymbol{p}^* = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

gives

$$\boldsymbol{p}* = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

as a solution.

Then

$$\boldsymbol{y}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^x, \qquad \boldsymbol{y}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^x + \begin{pmatrix} 5 \\ 2 \end{pmatrix} xe^x$$

are linearly-independent solutions and so the general solution becomes

$$c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^x + c_2 \left(\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} x \right) e^x.$$

Where does the second solution come from?

In $\boldsymbol{z}$ co-ordinates we have

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} d & 1 \\ 0 & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

i.e.,

$$z_1' = dz_1 + z_2 \qquad z_2' = dz_2.$$

The second equation is a stand-alone equation and has a solution $z_2 = e^{dx}$.

Therefore, from the first equation,

$$z_1' - dz_1 = e^{dx}.$$

This is a generic non-homogeneous first-order equation. Using the integrating factor approach we find it has a solution

$$z_1 = xe^{dx}.$$

Hence, translating back to $\boldsymbol{y}$ co-ordinates gives

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 5 & \frac{1}{2} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & \frac{1}{2} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} e^{dx} = \left(\begin{pmatrix} 5 \\ 2 \end{pmatrix} x + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right) e^{dx} \end{aligned}$$

as our second particular solution.

Finally, to verify, again calculate the Wronskian:

$$W = \begin{vmatrix} 5e^x & \frac{1}{2}e^x + 5xe^x \\ 2e^x & 2xe^x \end{vmatrix} = 10xe^{2x} - 2e^{2x}\left(\frac{1}{2} + 5x\right) = -e^{2x} \neq 0$$

Non-homogeneous equations

We now consider the case where

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{b}(x)$$

for non-zero $\boldsymbol{b}(x)$.

We know how to find a solution for the complementary homogeneous problem. Let $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$ denote the matrix that contains this solution. Note that

$$\boldsymbol{y}_1' = \boldsymbol{A}\boldsymbol{y}_1, \qquad \boldsymbol{y}_2' = \boldsymbol{A}\boldsymbol{y}_2, \qquad \text{etc.}$$

Hence,

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

for $\boldsymbol{Y}' = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')$.

We next look for a particular solution of the form $\boldsymbol{Y}\boldsymbol{u}$ for vector function $\boldsymbol{u}$.

This is the same technique as in the single-equation case.

If $\boldsymbol{y}_p = \boldsymbol{Y}\boldsymbol{u}$ is a solution then

$$\boldsymbol{y}_p' = \boldsymbol{Y}'\boldsymbol{u} + \boldsymbol{Y}\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{Y}\boldsymbol{u} + \boldsymbol{Y}\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{y}_p + \boldsymbol{Y}\boldsymbol{u}'.$$

If this is to be a solution to our system then

$$\boldsymbol{y}_p' = \boldsymbol{A}\boldsymbol{y}_p + \boldsymbol{b}(x)$$

must hold; this, of course, then means that

$$\boldsymbol{Y}\boldsymbol{u}' = \boldsymbol{b}(x)$$

must be true.

This can be solved by first finding $\boldsymbol{u}'$ and then integrating the result.

An example

Take

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4x}.$$

We first look for the two fundamental solutions to the complementary system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The characteristic polynomial is

$$\left| \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \left| \begin{pmatrix} 1-d & 2 \\ 2 & 1-d \end{pmatrix} \right| = (d-1)^2 - 4 = 0$$

and has two distinct solutions, $d_1 = 3$ and $d_2 = -1$.

The corresponding eigenvectors are

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \boldsymbol{p}_1 = \boldsymbol{0} \longrightarrow \boldsymbol{p}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \boldsymbol{p}_2 = \boldsymbol{0} \longrightarrow \boldsymbol{p}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The fundamental solutions are thus

$$\boldsymbol{y}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x}, \qquad \boldsymbol{y}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-x}.$$

Thus we have obtained

$$\boldsymbol{Y} = \begin{pmatrix} 1 & e^{-4x} \\ 1 & -e^{-4x} \end{pmatrix} e^{3x}, \qquad \boldsymbol{b}(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4x}$$

Now, we need to solve $\boldsymbol{Y}\boldsymbol{u}' = \boldsymbol{b}(x)$, that is,

$$\begin{pmatrix} 1 & e^{-4x} \\ 1 & -e^{-4x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{x}.$$

We have

$$\begin{pmatrix} 1 & e^{-4x} \\ 1 & -e^{-4x} \end{pmatrix}^{-1} = -\frac{e^{4x}}{2} \begin{pmatrix} -e^{-4x} & -e^{-4x} \\ -1 & 1 \end{pmatrix}$$

and so

$$\begin{aligned} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} 1 & e^{-4x} \\ 1 & -e^{-4x} \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{x} \\ &= \begin{pmatrix} e^{-4x} & e^{-4x} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \frac{e^{5x}}{2} \\ &= \begin{pmatrix} 3e^{-4x} \\ 1 \end{pmatrix} \frac{e^{5x}}{2} \\ &= \begin{pmatrix} \frac{3}{2}e^{x} \\ \frac{1}{2}e^{5x} \end{pmatrix} \end{aligned}$$

We can integrate both equations easily to get

$$u_1 = \frac{3}{2}e^{x}, \qquad u_2 = \frac{1}{10}e^{5x}.$$

Now it only remains to construct

$$\boldsymbol{y}_p = \boldsymbol{Y}\boldsymbol{u} = \begin{pmatrix} 1 & e^{-4x} \\ 1 & -e^{-4x} \end{pmatrix} e^{3x} \begin{pmatrix} \frac{3}{2}e^{x} \\ \frac{1}{10}e^{5x} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8 \\ 7 \end{pmatrix} e^{4x}$$

as a particular solution to the non-homogeneous problem.

Putting everything together we get

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + \boldsymbol{y}_p$$

as the general solution to our non-homogeneous system.

The autonomous 2-by-2 system

$$y_1' = a_{11}y_1 + a_{12}y_2$$
$$y_2' = a_{21}y_1 + a_{22}y_2$$

has critical point $(0,0)$. Assume here that it is the only critical point (So the coefficient matrix is invertible).

Its general solution is

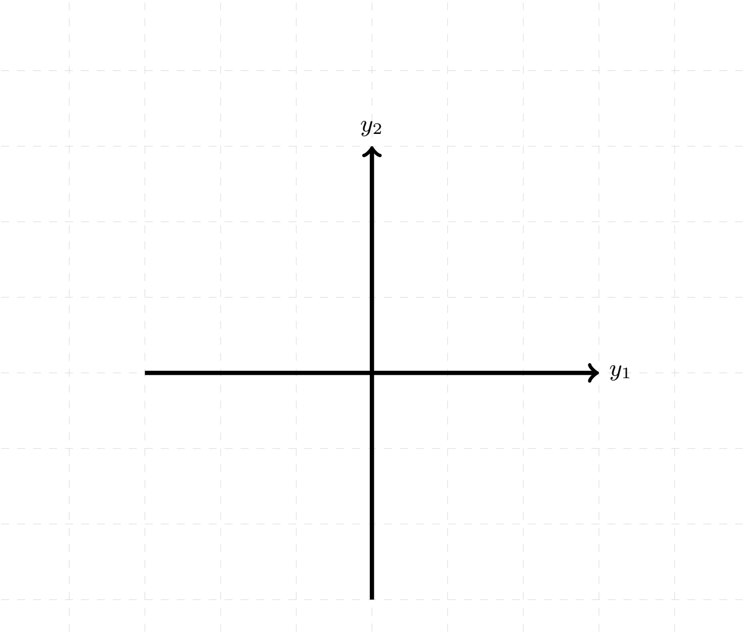
$$y_1 = c_1\, p_{11}\, e^{d_1 x} + c_2\, p_{12}\, e^{d_2 x}$$
$$y_2 = c_1\, p_{21}\, e^{d_1 x} + c_2\, p_{22}\, e^{d_2 x}$$

when $d_1 \neq d_2$

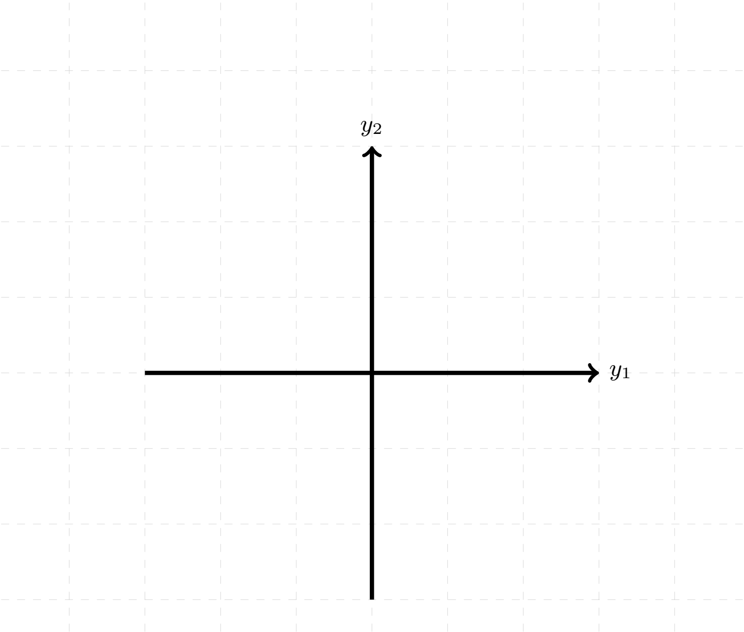
Consider the evolution of this system as $x \uparrow \infty$.

This depends on whether d_1 and d_2 are (real and) of the same sign.

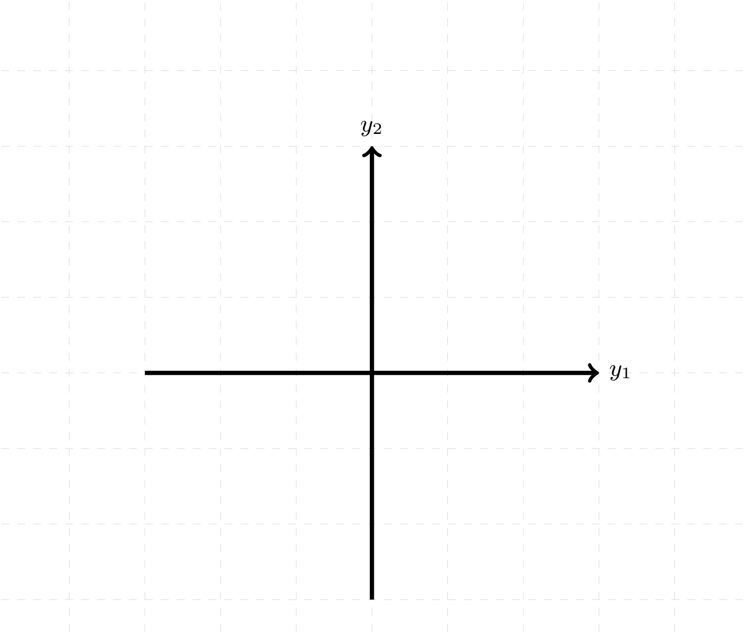
Node: $d_1 < d_2 < 0$



Node: $d_1 > d_2 > 0$



Saddle point: $d_1 < 0 < d_2$



Repeated root

Still consider

$$y_1' = a_{11}y_1 + a_{12}y_2$$
$$y_2' = a_{21}y_1 + a_{22}y_2.$$

If $a_{11} = a_{22} = a$ and $a_{12} = a_{21} = 0$ this is the system of stand-alone equations with solution

$$y_1 = c_1\, ae^{dx}$$
$$y_2 = c_2\, ae^{dx}.$$

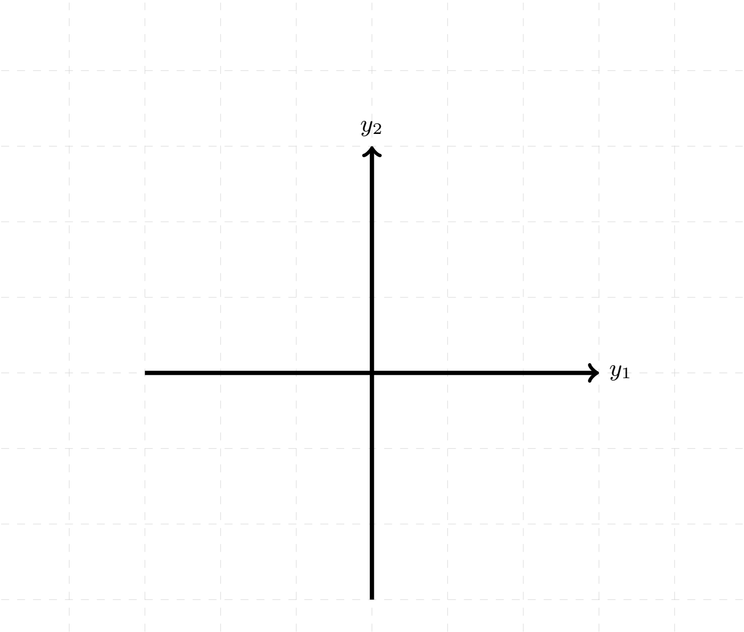
Otherwise

$$y_1 = c_1\, p_{11}\, e^{dx} + c_2\, (p_{12}^* + p_{11}x)\, e^{dx}$$
$$y_2 = c_1\, p_{21}\, e^{dx} + c_2\, (p_{22}^* + p_{21}x)\, e^{dx}$$

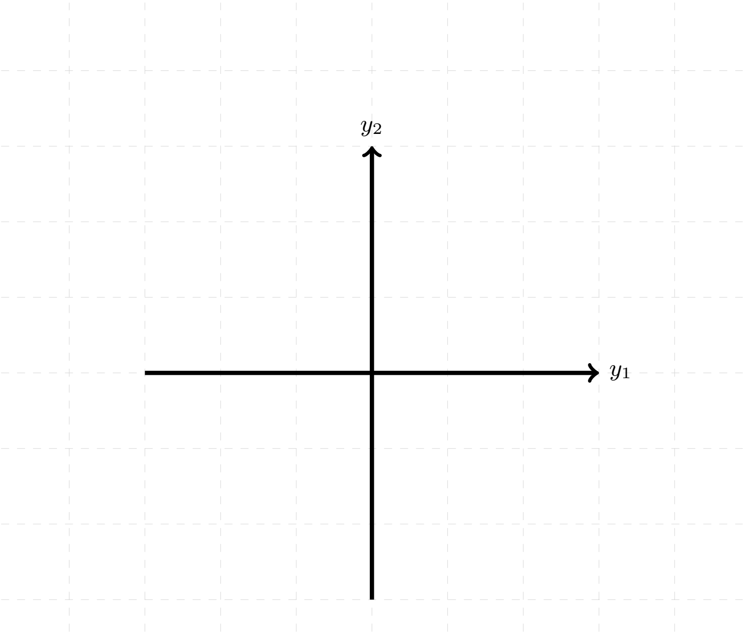
as per the above.

These cases give different phase planes.

Node: $d < 0$ (stand alone case)



Node: $d < 0$ (general case)



An exercise

The system

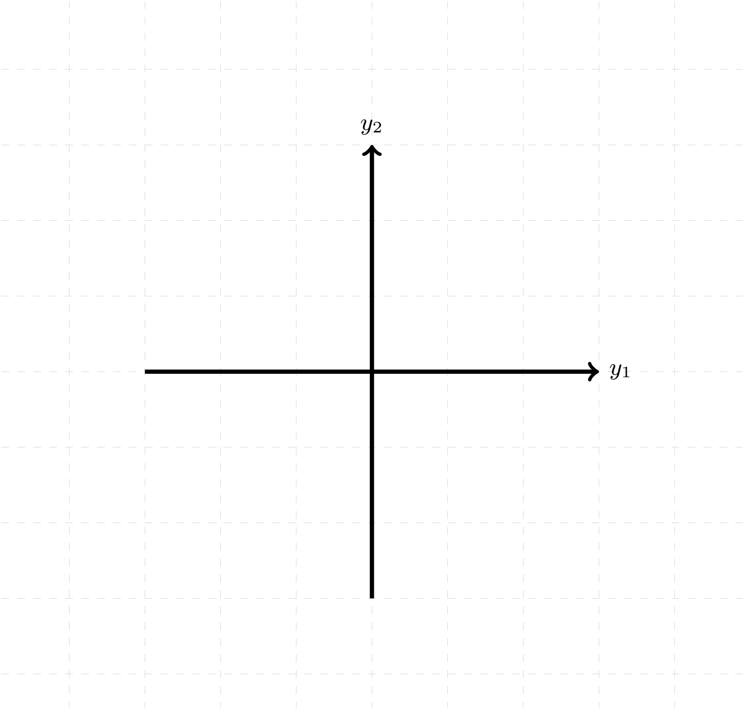
$$y_1' = y_1$$
$$y_2' = 2y_2 - y_1$$

has stationary point $(0, 0)$.

Show that its solution is

$$y_1 = c_1 e^x$$
$$y_2 = c_1 e^x + c_2 e^{2x}$$

Consider the evolution of this system as $x \uparrow \infty$ by plotting the phase plane.


y_2
y_1

Outline: Difference equations

First-order equations

Second-order equations

Higher-order systems

Notation

We consider a process at time $t = -(n-1), 1, 2, \ldots$.

We write $y_t = y(t)$.

A first-order equation is

$$y_t = ay_{t-1} + b$$

for constant coefficients a, b.

An nth-order equation is

$$y_t = a_1 y_{t-1} + \cdots a_n y_{t-n} + b.$$

The first $n-1$ observations are initial conditions.

Consider backward substitution of the process

$$y_t = ay_{t-1} + b.$$

We have

$$\begin{aligned}
&y_0\\
&y_1 = ay_0 + b\\
&y_2 = ay_1 + b = a^2 y_0 + ab + b\\
&y_3 = ay_2 + b = a^3 y_0 + a^2 b + ab + b\\
&\ \vdots
\end{aligned}$$

and so

$$y_t = a^t y_0 + b \sum_{j=0}^{t-1} a^j = a^t y_0 + b \frac{1-a^t}{1-a}.$$

As $t \to \infty$

$$y_t \to \frac{b}{1-a} = y^*$$

if $|a| < 1$; this is the stability condition. The limit is the steady-state solution; it is independent of the initial condition.

When $a = 1$, we have

$$y_0, \qquad y_1 = y_0 + b, \qquad y_2 = y_0 + 2b, \qquad y_t = y_0 + tb,$$

and y_t is a linear function of time.

When $a = -1$, we have

$$y_0, \qquad y_1 = -y_0 + b, \qquad y_2 = y_0, \qquad y_t = (-1)^t y_0 + b\,\{t \text{ is uneven}\},$$

which oscillates.

The process is explosive if $|a| > 1$.

Can also look at more general case

$$y_t = ay_{t-1} + b_t.$$

We do the same; backward substitution gives

$$\begin{aligned} y_t &= a\left(ay_{t-2} + b_{t-1}\right) + b_t \\ &= a^2 y_{t-2} + b_t + ab_{t-1} \\ &= a^2\left(ay_{t-3} + b_{t-2}\right) + b_t + ab_{t-1} \\ &= a^3 y_{t-3} + b_t + ab_{t-1} + a^2 b_{t-2} \\ &\vdots \\ &= a^t y_0 + \sum_{j=0}^{t-1} a^j\, b_{t-j}. \end{aligned}$$

Here,

$$\frac{dy_t}{db_{t-j}} = a^j.$$

Stick to the case where $b_t = b$ from now on.

In the stable homogeneous case

$$y_t = a y_{t-1}$$

the solution thus is $y_t = y_0 a^t$.

In the stable non-homogeneous case

$$y_t = a y_{t-1} + b$$

we convert to the homogeneous system by writing everything in deviations from the steady state:

$$\begin{aligned} y_t - y^* &= a(y_{t-1} - y^*) + (1-a)y^* + b \\ &= a(y_{t-1} - y^*) + (1-a)\frac{b}{1-a} + (1-a)\frac{b}{1-a} \\ &= a(y_{t-1} - y^*) \end{aligned}$$

and the solution for this then is $(y_t - y^*) = (y_0 - y^*)a^t$ or

$$y_t = y^* + (y_0 - y^*)a^t.$$

Second-order equations

For

$$y_t = a_1 y_{t-1} + a_2 y_{t-2}$$

a guess solution is of the form $y_t = c\alpha^t$.

Plugging this in shows that

$$c\alpha^t - a_1 c\alpha^{t-1} - a_2 c\alpha^{t-2} = c\alpha^{t-2}(\alpha^2 - a_1\alpha - a_2) = 0$$

must hold at the solution.

Equivalently,

$$\alpha^2 - a_1\alpha - a_2 = 0,$$

which has two solutions in general:

$$\alpha_1 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2}, \qquad \alpha_2 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2},$$

so

$$y_t = c_1\alpha_1^t + c_2\alpha_2^t.$$

The initial conditions (i.e., y_{-1} and y_0) pin down c_1, c_2.

Higher-order equations

The nth-order equation

$$y_t = a_1 y_{t-1} + \cdots + a_n y_{t-(n-1)} + b$$

can be written as

$$\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-(n-1)} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-n} \end{pmatrix} + \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or, in matrix form

$$\boldsymbol{y}_t = \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{b},$$

which is a first-order vector system.

Generic system of equations has the same form, for suitable matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$.

The steady-state here will be

$$\boldsymbol{y}^* = (\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}$$

provided the inversion is justified.

In a homogeneous system

$$\boldsymbol{y}_1 = \boldsymbol{A}\boldsymbol{y}_0, \qquad \boldsymbol{y}_2 = \boldsymbol{A}\boldsymbol{y}_1 = \boldsymbol{A}\boldsymbol{y}_0 = \boldsymbol{A}^2\boldsymbol{y}_0, \cdots, \qquad \boldsymbol{y}_t = \boldsymbol{A}^t\boldsymbol{y}_0$$

The solution for the non-homogeneous case then is

$$\boldsymbol{y}_t = \boldsymbol{y}^* + \boldsymbol{A}^t(\boldsymbol{y}_0 - \boldsymbol{y}^*)$$

The stability condition here is the requirement that all eigenvalues of $\boldsymbol{A}$ are smaller than one in magnitude.

Can link this back to the approach taken for differential equations.

Say $\boldsymbol{A}$ is diagonalizable, i.e.,

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

for diagonal matrix of eigenvalues $\boldsymbol{D}$.

Then

$$\boldsymbol{z} = \boldsymbol{P}^{-1}\boldsymbol{y}$$

satisfies

$$\boldsymbol{z}_t = \boldsymbol{z}^* + \boldsymbol{D}^t(\boldsymbol{z}_0 - \boldsymbol{z}^*),$$

in obvious notation.

This follows immediately from substitution, using that $\boldsymbol{A}^t = \boldsymbol{P}\boldsymbol{D}^t\boldsymbol{P}^{-1}$.

An example

For $n = 2$ the system is

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ solve

$$\begin{vmatrix} a_1 - \alpha & a_2 \\ 1 & -\alpha \end{vmatrix} = \alpha(\alpha - a_1) - a_2 = \alpha^2 - a_1\alpha - a_2 = 0,$$

which is what we found before.